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Translated by N. H. C.

UDC 62-50

ESCAPE OF NONLINEAR OBJECTS OF DIFFERENT TYPES WITH INTEGRAL CONSTRAINTS ON THE CONTROL

PMM Vol. 39, №1, 1975, pp. 12-23
P. B. GUSIATNIKOV
(Moscow)
(Received December 24, 1973)

We quote sufficient conditions for evasion of contact in a game of two nonlinear objects with integral constraints on the control.

1. Let t_0 be a fixed real number. Let the laws of motion of the pursuing vector $x \in E^n$ and of the escaping vector $y \in E^n$ be described for $t \geqslant t_0$ by the vector differential equations

$$\begin{aligned} d^kx / dt^k &= L(t, X) + u, \quad x = \operatorname{col}(x^1, \ldots, x^n), \qquad u = u(t) \in E^n \\ X &= \operatorname{col}\{x^{(0)}, x^{(1)}, \ldots, x^{(k-1)}\}; \; x^{(i)} = d^ix / dt^i, \qquad 0 \leqslant i \leqslant k-1 \\ L(t, X) &= L(t, x^{(0)1}, \ldots, x^{(0)n}, x^{(1)1}, \ldots, x^{(k-1)n}) \end{aligned} \tag{1.1}$$

$$\begin{aligned} d^ly / dt^l &= H(t, Y) + v, \quad y = \operatorname{col}(y^1, \ldots, y^n), \qquad v = v(t) \in E \\ Y &= \operatorname{col}\{y^{(0)}, \ldots, y^{(l-1)}\}; \; y^{(j)} = d^jy / dt^j, \qquad 0 \leqslant j \leqslant l-1 \\ H(t, Y) &= H(t, y^{(0)1}, \ldots, y^{(0)n}, y^{(1)1}, \ldots, y^{(l-1)n}) \end{aligned} \tag{1.2}$$

Here E^n is an n-dimensional Euclidean space, $u(v)$ is an everywhere finite vector-valued function, measurable for $t \geqslant t_0$, whose scalar square we sum on any interval $[t_1, t_2] \subset [t_0, +\infty]$, called the control of the pursuer (escaper), $X(Y)$ is the phase vector of the pursuer (escaper), $L(t, X)$, $H(t, Y)$ are vector-valued functions continuous together with their first-order partial derivatives in all variables.

We assume that the following condition is satisfied for game (1.1), (1.2): for arbitrary collection $z_* = \{t_*, X_*, Y_*\}$, $t_* \geqslant t_0$, called the (initial) point of the game, and for arbitrary players' controls, the solutions $X(t)$ and $Y(t)$ of Eqs. (1.1) and (1.2), respectively, in the sense of Carathéodory [1], with initial values $X(t_*) = X_*$, $Y(t_*) = Y_*$, exist on the whole interval $[t_*, +\infty]$.

The following constraints are imposed on the players' controls:

$$\int_{t_0}^{+\infty} \rho(t, X(t))(u(t) \cdot u(t))\, dt \leqslant \rho^2 \tag{1.3}$$

$$\int_{t_0}^{+\infty} \sigma(t, Y(t))(v(t) \cdot v(t))\, dt \leqslant \sigma^2 \tag{1.4}$$

where $\rho(t, X) > 0$ and $\sigma(t, Y) > 0$ are scalar functions continuously differentiable in all the arguments, $\rho > 0$ and $\sigma > 0$ are fixed constants.

For objects described by conditions (1.1) – (1.4) we examine the evasion-of-contact problem (see [2, 3]): in accordance with the information known to the escaper, at each instant t choose the control vector $v(t)$ so that the equality $x(t) = y(t)$ is not satisfied for any finite t. It is assumed that at each instant t the escaper knows the point $z(t) = \{t, X(t), Y(t)\}$ of the game and the vector $u(t)$.

2. We say that the escaper has a maneuver superiority over the pursuer if one of the following two conditions are satisfied: (1) $l < k$; (2) $l = k$

$$\sigma > \rho \tag{2.1}$$

$$\sigma(t, y^{(0)}, \ldots, y^{(l-1)}) \leqslant \rho(t, x^{(0)}, \ldots, x^{(k-1)}) \tag{2.2}$$

as soon as $x^{(0)} = y^{(0)}$.

Theorem on evasion of contact. If the escaper has a maneuver superiority over the pursuer, then evasion of contact is possible. Here, for arbitrary initial game point $z_0 = \{t_0, X_0, Y_0\}$ satisfying the condition $x_0^{(0)} \neq y_0^{(0)}$, by a suitable choice of escape control $v = \bar{v}(t)$ we can ensure the following estimate on the distance $\xi(t) = |\psi(t)|$, $\psi(t) = y(t) - x(t)$ between the players:

$$\xi(t) \geqslant \begin{cases} \varepsilon_2 \gamma(\eta(t)) (\xi(t_0))^l, & t_0 \leqslant t \leqslant T_2 \quad \text{for} \quad \xi(t_0) < \varepsilon \\ \varepsilon_2 \gamma(\eta(t)) \varepsilon^l, & t_0 \leqslant t \leqslant T_2 \quad \text{for} \quad \xi(t_0) \geqslant \varepsilon \\ \varepsilon_m \gamma(\eta(t)), & T_{m-1} \leqslant t \leqslant T_m; \quad m = 3, 4, \ldots, (T_m \to \infty, m \to \infty) \end{cases} \tag{2.3}$$

$$\eta(t) = \eta(t, X, Y) = [1 + t^2 + |x^{(0)}|^2 + \ldots + |x^{(k-1)}|^2 + |y^{(0)}|^2 + \ldots + |y^{(l-1)}|^2]^{1/2}$$

Here $\varepsilon, \varepsilon_2, \varepsilon_3, \ldots$ are positive numbers and $\gamma(\eta)$ is a monotonically decreasing function of its argument, they depend solely on problem (1.1) – (1.4) and are independent of the initial values of the players' phase coordinates or of the progress of the game; $t_0 < T_2 < T_3 < \ldots$ is a sequence which depends not only on problem (1.1) – (1.4) but also on the evasion process.

Following [2] we prove the theorem in several argument stages. By $(a \cdot b) = a^1 b^1 + \ldots + a^n b^n$ we denote the scalar product of the vectors $a \in E^n$ and $b \in E^n$, $|a| = (a \cdot a)^{1/2}$.

3. By virtue of system (1.1), (1.2), for an arbitrary function $\eta = \eta(t)$ we have the estimate

$$|\eta \cdot \eta^\cdot| = |t + (x \cdot x^\cdot) + \ldots + (x^{(k-1)} \cdot L(t, X) + u) + (y \cdot y^\cdot) + \ldots + (y^{(l-1)} \cdot H(t, Y) + v)| \leqslant \eta^2 + \eta(|L(t, X)| + |H(t, Y)| + |u| + |v|)$$

Hence

$$|\eta^\cdot| \leqslant \eta + c^*(\eta) + |u| + |v| \leqslant \lambda(\eta) + |u| + |v| \tag{3.1}$$

where

$$c^*(r) = \sup_{\eta(t, X, Y) \leqslant r} \{|L(t, X)| + |H(t, Y)| + \sigma(t, Y)\}$$

$$\lambda(r) = 1 + r + r^2 + c^*(r)$$

(here the sup in the right-hand side is taken over all possible collections $z = \{t, X, Y\}$ of phase coordinates, satisfying the constraints $t \geqslant t_0$, $\eta(t, X, Y) \leqslant r$).

We set

$$\varepsilon^*(r) = \min\{1, \min_{\eta(t,X,Y)\leqslant r} (\rho(t, X))^{1/2}, \min_{\eta(t,X,Y)\leqslant r} (\sigma(t, Y))^{1/2}\}$$

$$F(r) = \int_0^r \varepsilon^*(s)(\lambda(s))^{-1}\, ds$$

From the continuity of functions $\varepsilon^*(s) > 0$ and $c^*(s) \geqslant 0$ it follows that the function $F(r)$ is defined and is strictly monotonic on the half-interval $[0 + \infty)$, so that the function $\Phi(r)$ inverse to it grows, also strictly monotonically, on the half-interval $[0, \alpha^*)$, where $\alpha^* = \lim_{r\to+\infty} F(r) < +\infty$. From relation (3.1) it follows that the estimate

$$\left|\frac{dF(\eta(t))}{dt}\right| \leqslant \varepsilon^*(\eta(t))\left(1 + \frac{|u(t)| + |v(t)|}{\lambda(\eta(t))}\right) \leqslant$$
$$1 + (\rho(t, X(t)))^{1/2}|u(t)| + (\sigma(t, Y(t)))^{1/2}|v(t)|$$

($\varepsilon^*(r) \leqslant 1 \leqslant \lambda(r)$) is valid for the phase vectors of system (1.1), (1.2) with any initial condition $\{t_*, X_*, Y_*\}$, $t_* \geqslant t_0$, and with any controls $u(t)$ and $v(t)$, $t_* \leqslant t < \infty$. So that when $|t - t_*| \leqslant 1$, by virtue of (1.3), (1.4)

$$|F(\eta(t)) - F(\eta(t_*))| \leqslant (t - t_*) + (\rho + \sigma)(t - t_*)^{1/2} \leqslant$$
$$\varphi^*(t - t_*)$$
$$\varphi^*(r) = ar^{1/2}, \quad a = 1 + \rho + \sigma$$

and, consequently,

$$\Phi(F(\eta_*) - \varphi^*(t - t_*)) \leqslant \eta(t) \leqslant \Phi(F(\eta_*) + \varphi^*(t - t_*)), \quad t \in [t_*, t_* + \theta_*] \tag{3.2}$$
$$\eta_* = \eta(t_*), \quad \theta_* = \min\{1, 4\delta^*(\eta_*), \chi(\eta_*)\}$$
$$\delta^*(r) = \left[\frac{\alpha^* - F(r)}{2a}\right]^2, \quad \chi(r) = \frac{F^2(r)}{a^2}$$

Let $A = A(w) \in E^n$ be a differentiable vector-valued function of the variable $w \in E^n$ and let b be an arbitrary vector from E^n. By the product $(\partial A / \partial w \cdot b)$ we mean a vector from E^n, each of whose components is a scalar product of the gradient of the corresponding component of the vector-valued function A by vector b. Then, the estimate

$$\left|\frac{dL(t, X(t))}{dt}\right| + \left|\frac{dH(t, Y(t))}{dt}\right| \leqslant R(t) \tag{3.3}$$

is obvious for the solutions $X(t)$ and $Y(t)$ of system (1.1), (1.2). Here (the sup is taken over all $\eta(t, X, Y) \leqslant r$; $f, w \in E$, $|f| = |w| = 1$)

$$\mu_1(r) = \sup\{p(t, X) + q(t, Y)\} + \lambda(r) + \sup\left|\left(\frac{\partial\sigma(t, Y)}{\partial y^{(0)}} \cdot f\right)\frac{1}{\rho(t, X)}\right|$$

$$p(t, X) = \left|\frac{\partial L(t, X)}{\partial t} + \sum_{i=0}^{k-2}\left(\frac{\partial L(t, X)}{\partial x^{(i)}} \cdot x^{(i+1)}\right) + \left(\frac{\partial L(t, X)}{\partial x^{(k-1)}} \cdot L(t, X)\right)\right|$$

$$q(t, Y) = \left|\frac{\partial H(t, Y)}{\partial t} + \sum_{j=0}^{l-2}\left(\frac{\partial H(t, Y)}{\partial y^{(j)}} \cdot y^{(j+1)}\right) + \left(\frac{\partial H(t, Y)}{\partial y^{(l-1)}} \cdot H(t, Y)\right)\right|$$

$$\mu_2(r) = \sup\left\{\left|\left(\frac{\partial L(t, X)}{\partial x^{(k-1)}} \cdot w\right)(\rho(t, X))^{-1/2}\right| + \left|\left(\frac{\partial H(t, Y)}{\partial y^{(l-1)}} \cdot f\right)(\sigma(t, Y))^{-1/2}\right|\right\}$$

$$\begin{aligned}\Gamma(r) &= \max\{\mu_1(r),\ (\mu_2(r))\}\\ R(t) &= \Gamma(\eta(t)) + \Gamma(\eta(t))\{(\rho(t,\ X(t)))^{1/2}\,|\,u(t)\,| +\\ &\quad (\sigma(t,\ Y(t)))^{1/2}\,|\,v(t)\,|\,\}\end{aligned} \qquad (3.4)$$

If $l < k$, it is easily verified that

$$\left|\frac{dH(t, Y(t))}{dt}\right| + |x^{(l+1)}(t)| \leqslant R(t) \qquad (3.5)$$

4. Let us introduce the notion of a special control for the escaper. At first we examine the case $l = k$. Let $c \in (0,\ (3\cdot 2^l)^{-1})$ be some constant and $\omega \in E^n$ be an arbitrary vector satisfying the constraint

$$|\,\omega\,| \leqslant c \qquad (4.1)$$

Then, for arbitrary control $u(s)$, $s \geqslant t_0$, there exists a control $v_{\omega,c}(s)$ such that

$$v_{\omega,c}(s) = u(s) + l!\,\omega \qquad (4.2)$$

Let $t_* \geqslant t_0$ and $t > t_*$ be the arbitrary real numbers. Multiplying (4.2) by $(t-s)^{l-1}$ and integrating from t_* to t, we obtain (here and subsequently $\tau = t - t_*$)

$$\frac{1}{(l-1)!}\int_{t_*}^{t}(t-s)^{l-1}\,[v_{\omega,c}(s) - u(s)]\,ds = \omega\tau^l \qquad (4.3)$$

In the case $l < k$ the escaper's special control is given by the formula

$$v_{\omega,c}(s) = \omega l! \qquad (4.4)$$

Then

$$\frac{1}{(l-1)!}\int_{t_*}^{t}(t-s)^{l-1}\,v_{\omega,c}(s)\,ds = \omega\tau^l \qquad (4.5)$$

In Sects. 5 – 7 we describe the escaper's active behavior.

5. Let the game (1.1) – (1.4) commence at the point

$$z_* = \{t_*, X_*, Y_*\},\quad t_* \geqslant t_0,\quad 0 < |x_*^{(0)} - y_*^{(0)}| = \xi(t_*) < 1$$

and, for given $c > 0$ and $\omega \in E$ let it develop under the action of the special control $v_{\omega,c}(s)$. Then, for $t \geqslant t_*$ the vector-valued function $\psi(t)$ can be written in the following form:

$$\psi(t) = T_*(\tau) + \frac{1}{(l-1)!}\int_{t_*}^{t}(t-s)^{l-1}\,[y^{(l)}(s) - x^{(l)}(s)]\,ds \qquad (5.1)$$

$$T_*(\tau) = \psi(t_*) + \sum_{j=1}^{l-1} d_j\tau^j;\quad d_j = (j!)^{-1}(y_*^{(j)} - x_*^{(j)}),\quad 1 \leqslant j \leqslant l-1$$

Formula (5.1) is obtained by a Taylor series expansion of function $\psi(t)$, with a remainder term in integral form. Substituting $x^{(l)}(s)$ and $y^{(l)}(s)$ from relations (1.1), (1.2) and integrating by parts, we obtain (see (4.3), (4.5))

$$\psi(t) = T(\tau) + \omega\tau^l + h(t)\,\tau^l,\quad T(\tau) = T_*(\tau) + d_l\tau^l \qquad (5.2)$$

$$l!\,d_l = \begin{cases} H(t_*, Y_*) - L(t_*, X_*), & l = k\\ H(t_*, Y_*) - x_*^{(l)}, & l < k\end{cases}$$

$$l!\ \tau^l h(t) = \begin{cases} \int\limits_{t_*}^{t} (t-s)^l \left[\frac{d}{ds}(H(s,Y(s)) - L(s,X(s)))\right] ds, & l = k \\ \int\limits_{t_*}^{t} (t-s)^l \left[\frac{d}{ds} H(s,Y(s)) - x^{(l+1)}(s)\right] ds, & l < k \end{cases}$$

Although the vector-valued function $h(t)$ depends upon the players controls, nevertheless it satisfies (see (1.3), (1.4), (3.2) – (3.5)) the constraint

$$|h(t)| \leqslant \int\limits_{t_*}^{t} R(s)\, ds \leqslant \alpha(\eta_*, \tau)\, \tau^{1/2}, \quad 0 \leqslant \tau \leqslant \theta_* \tag{5.3}$$

Here

$$\alpha(\eta_*, \tau) = a\Gamma(\Phi(F(\eta_*) + \varphi^*(\tau))) \tag{5.4}$$

In addition, as can be verified,

$$|d_j| \leqslant 2\eta_* \leqslant \lambda(\eta_*) \leqslant \Gamma(\eta_*), \quad 1 \leqslant j \leqslant l-1 \tag{5.5}$$
$$|d_l| \leqslant c^*(\eta_*) + \eta_* \leqslant \lambda(\eta_*) \leqslant \Gamma(\eta_*)$$

6. If the estimate (2.3) is fulfilled for the orthogonal projection of curve (5.2) onto some two-dimensional subspace of space E^n, then, obviously, it is fulfilled for curve (5.2) itself as well. Therefore, without loss of generality we can take E^n as being two-dimensional. In E^n we choose an orthonormalized basis such that the vector $\psi(t_*)$ has the components $(\xi_*, 0)$ in it. In this basis let

$$d_j = (d_j^1, d_j^2),\ j = 1, \ldots, l; \qquad \omega = (\omega^1, \omega^2)$$
$$h(t) = (h^1(t), h^2(t)),\ T(\tau) = (T^1(\tau), T^2(\tau))$$

Then the equation for curve (5.2) is rewritten as follows:

$$\psi^1(t) = \xi_* + \sum_{j=1}^{l} d_j^1 \tau^j + (\omega^1 + h^1(t))\,\tau^l \tag{6.1}$$

$$\psi^2(t) = \sum_{j=1}^{l} d_j^2 \tau^j + (\omega^2 + h^2(t))\,\tau^l$$

Let us set (here and in further cases the dependency on z_* is not explicitly noted) $g(c) = (6/c)^{1/l}$, $\varepsilon(c) \leqslant (c/6)^{2l+1}$ (the final choice of $\varepsilon(c)$ is made in Sect. 7). By $\tau = \tau(c) \in (0, \delta^*(\eta_*))$ we denote the solution of the equation

$$\beta(\tau) \equiv \tau - g(c)(\varepsilon(c))^{1/l} \{1 + \delta^{*l-1}(\eta_*) + \alpha^2(\eta_*, \tau) + \chi^{-1}(\eta_*)\}^{-1} = 0 \tag{6.2}$$

existing because $\beta(\tau)$ is continuous and $\beta(0) < 0$ and $\beta(\delta^*(\eta_*)) > 0$. We denote the number $t_* + \tau(c)$ by t^*

Let us prescribe that on the interval $[t_*, t^*]$ the escaper applies the special control $v_{\omega, c}(s)$ such that

$$\omega^1 \in [{}^1/_2 c, {}^3/_4 c],\ T^1(\tau(c)) \geqslant 0$$
$$\omega^1 \in [-{}^3/_4 c, -{}^1/_2 c],\ T^1(\tau(c)) < 0$$

(the final choice of ω is put aside until Sect. 7)). Then, as is easily verified,

$$|\psi^1(t^*)| \geqslant 2\varepsilon(c)\, p^{-l}(\eta_*) > \varepsilon(c)\, p^{-l}(\eta_*)$$

$$p(\eta_*) = 1 + \delta^{*-1}(\eta_*) + \alpha^2(\eta_*, \tau(c)) + \chi^{-1}(\eta_*)$$

Since (see (3.2))

$$F(\eta(t^*)) \geqslant F(\eta_*) - \varphi^*(\tau(c)) > F(\eta_*) - \frac{\alpha^* - F(\eta_*)}{2} \geqslant \frac{3F(\eta_*) - \alpha^*}{2}$$

and, consequently,

$$\frac{F(\eta(t^*)) + 2\alpha^*}{3} > F(\eta_*) + \varphi^*(\tau(c)) \tag{6.3}$$

we have

$$1 < p^l(\eta_*) < \pi(\eta(t^*))$$

$$\pi(r) = \left\{1 + \frac{9}{4}\delta^{*-1}(r) + \left(a\Gamma\left(\Phi\left(\frac{F(r) + 2\alpha^*}{3}\right)\right)\right)^2 + \chi^{-1}(1)\right\}^l$$

Hence

$$\xi(t^*) \geqslant |\psi^1(t^*)| > \frac{\varepsilon(c)}{\pi(\eta(t^*))} \tag{6.4}$$

The next stage of arguments is most essential in the proof of the evasion-of-contact theorem since it is precisely here that we determine definitively the quantity $\varepsilon(c)$ (and the quantity $\tau(c)$) and make the final choice of the vector ω on the interval $[t_*, t^*]$.

7. By $\rho(t)$ and $\varphi(t)$ we denote the polar coordinates of a point $\psi(t)$ of curve (5.2). We set

$$\alpha = \omega^1 + h^1(t),\ \beta = \omega^2 + h^2(t) \tag{7.1}$$

Multiplying relations (6.1) in succession by $1, \tau, \ldots, \tau^{l-1}$, we obtain

$$\rho(t)\cos\varphi(t) = \xi_* + \sum_{j=1}^{l} a_j^1\tau^j + \alpha\tau^l \tag{7.2}$$

$$\tau\rho(t)\cos\varphi(t) = \tau\xi_* + \sum_{j=1}^{l} d_j^1\tau^{j+1} + \alpha\tau^{l+1}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\tau^{l-1}\rho(t)\sin\varphi(t) = \sum_{j=1}^{l} d_j^2\tau^{j+l-1} + \beta\tau^{2l-1}$$

We treat these relations as a system of $2l$ linear algebraic equations in the unknowns $1, \tau, \ldots, \tau^{2l-1}$. Solving it formally for the unknown unity, we find

$$1 \cdot D = D_1$$

where D is the system's determinant, while D_1 is the determinant resulting from D by replacing the first column by the column of free terms of system (7.2). From the first column of D_1 we take out the common factor $\rho(t)$ and we set $D_1 = \rho(t)D^*$. Then we obtain

$$\rho(t)D^* = D \tag{7.3}$$

The estimate (see (5.5))

$$|D^*| \leqslant (2l)!\ |\Gamma(\eta_*)|^{2l-1} \tag{7.4}$$

for D^* is obvious.

The determinant $D = D(\alpha, \beta)$ is a polynomial in α and β, with coefficients depending on point z_*. We see immediately that

$$D(\alpha, \beta) = p_{0l}\beta^l + \sum_{i,j=0}^{l-1} p_{ij}\alpha^i\beta^j \qquad (p_{0l} = \xi_*{}^l) \tag{7.5}$$

Let p be the coefficient of polynomial (7.5) largest in absolute value. Then

$$D(\alpha, \beta) = ps(\alpha, \beta), \ |p| \geqslant \xi_*{}^l \tag{7.6}$$

Here $s(\alpha, \beta)$ is a polynomial of form (7.5), all of whose coefficients do not exceed unity in absolute value, and one of the coefficients equals unity. We examine this polynomial on two rectangles

$$\Pi_1 = \{5c/8 \leqslant \alpha \leqslant 7c/8, \ -c/8, \leqslant \beta \leqslant c/8\}$$
$$\Pi_2 = \{-7c/8 \leqslant \alpha \leqslant -5c/8, \ -c/8 \leqslant \beta \leqslant c/8\}$$

The subsequent arguments are carried out for Π_1 because for Π_2 the arguments and estimates are absolutely identical.

Let $b \geqslant a > 0$, $\delta > 0$ and h be arbitrary real numbers, m be a positive integer. By $Q(a, b, h, \delta, m)$ we denote a family of mth-degree polynomials, to be examined on the interval $[h - \delta, h + \delta]$, all of whose coefficients do not exceed number b in absolute value, but such that the absolute value of at least one of the coefficients is greater than or equal to a.

Lemma 1. If $|h| < 1$ and $P(x) = p_0 + p_1x + \dots + p_mx^m \in Q(a, b, h, \delta, m)$, then

$$P^*(y) = P(y + h) \in Q(a - 2^m b\,|h|\,(1 - |h|)^{-1}, \ 2^m b\,(1 - |h|)^{-1}, \ 0, \ \delta, \ m)$$

Proof. After simplification we have

$$P^*(y) = \sum_{k=0}^{m} p_k{}^* y^k; \qquad p_k{}^* = \sum_{j=k}^{m} p_j C_j^k h^{j-k}, \qquad 0 \leqslant k \leqslant m \tag{7.7}$$

So that

$$|p_k{}^*| \leqslant b \cdot 2^m \sum_{j=k}^{m} |h|^{j-k} \leqslant 2^m b (1 - |h|)^{-1}, \quad 0 \leqslant k \leqslant m$$

If, however, k_0 is such that $|p_{k_0}| \geqslant a$, then by virtue of (7.7)

$$|p_{k_0}{}^*| \geqslant |p_{k_0}| - \sum_{j=k_0+1}^{m} |p_j|\,|C_i^{k_0}|\,|h|^{j-k_0} \geqslant a - 2^m b\,|h|\,(1 - |h|)^{-1}$$

Q.E.D.

Let $p(x)$ be a polynomial in x, to be examined on $[a^*, b^*]$. We set

$$\|p(x)\| = \max_{a^* \leqslant x \leqslant b^*} |p(x)|$$

Lemma 2. If $\delta < 1$ and $P(x) = p_0 + p_1x + \dots + p_mx^m \in Q(a, b, 0, \delta, m)$, then

$$\|P(x)\| \geqslant a\,(2^m m!)^{-1}\,\delta^m; \qquad a^* = -\delta, \ b^* = +\delta$$

Proof. Having set $x = \delta z$, we obtain $P(x) = P^*(z)$, where

$$P^*(z) = \sum_{k=0}^{m} (p_k \delta^k)\, z^k$$

Therefore, $P^*(z) \in Q(a\delta^m, b, 0, 1, m)$. Let $|p_{k_0}| \geqslant a$. Then

$$\| d^{k_0}P^*(z)/dz^{k_0} \| \geqslant | d^{k_0}P^*(0)/dz^{k_0} | = | p_{k_0} | \delta^{k_0} k_0! \geqslant k_0! \, \delta^m a$$

By virtue of the Markov inequality (see [4])

$$\left\| \frac{d^{k_0-1}P^*(z)}{dz^{k_0-1}} \right\| \geqslant (m-k_0+1)^{-2} \left\| \frac{d^{k_0}P^*(z)}{dz^{k_0}} \right\| \geqslant \frac{k_0! \, \delta^m a}{(m-k_0+1)^2}$$

Hence by induction

$$\| P^*(z) \| \geqslant k_0! \, \delta^m a \, ((m-k_0)!)^2 (m!)^{-2} \geqslant a \, (2^m m!)^{-1} \delta^m$$

Q.E.D.

The polynomial $s(\alpha, \beta)$ can be looked upon as a polynomial in β with coefficients depending on α

$$s(\alpha, \beta) = s_{0l}\beta^l + \sum_{j=0}^{l-1} \left(\sum_{i=0}^{l-1} s_{ij}\alpha^j \right) \beta^j$$

Obviously

$$\left| \sum_{i=0}^{l-1} s_{ij}\alpha^i \right| \leqslant \sum_{i=0}^{l-1} | \alpha^i | \leqslant 8 \quad \text{for} \quad |\alpha| \leqslant \frac{7c}{8} < \frac{7}{8} \qquad (7.8)$$

Let i_0 and j_0 be such that $s_{i_0 j_0} = 1$. Then

$$P(\alpha) = \sum_{i=0}^{l-1} s_{ij_0}\alpha^i \in Q\left(1, 1, \frac{3c}{4}, \frac{c}{8}, l-1\right)$$

Therefore, by virtue of Lemma 1, for $c < (3 \cdot 2^l)^{-1}$

$$P^*(\gamma) \in Q\,(1 - 3 \cdot 2^{l-1} c, \;\; 2^{l+2}, 0, \; c/8, \;\; l-1) \subset Q\,(1/2, \; 2^{l+2}, \; 0, \; c/8, \; l-1)$$

$$\gamma = \alpha - 3c/4, \; P^*(\gamma) = P(\gamma + 3c/4)$$

In accord with Lemma 2 we can then find γ_0, $|\gamma_0| \leqslant c/8$, such that

$$|P^*(\gamma_0)| \geqslant \lambda c^{l-1}, \;\; \lambda = (2^{4l-3}(l-1)!)^{-1} \qquad (7.9)$$

We fix $\alpha_0 = \gamma_0 + 3c/4 \in [5c/8, 7c/8]$. Then (see (7.8), (7.9))

$$s(\beta) = s(\alpha_0, \beta) \in Q\,(\lambda \cdot c^{l-1}, 8, 0, c/8, l)$$

So that by virtue of Lemma 2 we can find β_0, $|\beta_0| \leqslant c/8$, such that

$$| s(\alpha_0, \beta_0) | \geqslant | s(\beta_0) | \geqslant \Delta c^{2l-1}, \;\; \Delta = \lambda \, (4^{2l} \, l!)^{-1}$$

Since the partial derivatives of polynomial $s(\alpha, \beta)$ are bounded on the rectangle

$$\Pi_1^* = \{9c/16 \leqslant \alpha \leqslant 15c/16, \; -3c/16 \leqslant \beta \leqslant 3c/16\}$$

by a constant $\Delta_1 = 16 \cdot l \cdot 16/13$, we have

$$| s(\alpha_0 + \delta\alpha, \; \beta_0 + \delta\beta) | \geqslant {}^1/_2 \Delta \cdot c^{2l-1} = r(c) \qquad (7.10)$$

as soon as $|\delta\alpha| \leqslant \delta(c)$, $|\delta\beta| \leqslant \delta(c)$, where

$$\delta(c) = \Delta^* c^{2l-1}, \;\; \Delta^* = \min\{1/16, \; \Delta/(4\Delta_1)\}$$

We fix

$$\varepsilon(c) = \varepsilon^* c^{4l^2-2l+1}, \qquad \varepsilon^* = \min\{6^{-2l-1}, \; (\Delta^*)^{2l}/6\} \qquad (7.11)$$

(the inequality $\varepsilon(c) \leqslant (c/6)^{2l+1}$ is then guaranteed) and we prescribe that on the interval $[t_*, t^*]$ the escaper applies the special control $v_{\omega, c}(s)$, where $\omega = (\alpha_0, \beta_0)$, and (α_0, β_0) is a point of the Π_1, constructed during the arguments presented above, if $T^1(\tau(c)) \geqslant 0$, and (α_0, β_0) is a point of the Π_2, constructed absolutely analogously, if $T^1(\tau(c)) < 0$. Then by virtue of (5.3) and (7.10)

$$|h(t)| \leqslant \alpha(\eta_*, \tau(c))(\tau(c))^{1/2} \leqslant (g(c)(\varepsilon(c))^{1/l})^{1/2} \leqslant \delta(c) \tag{7.12}$$

So that (see (7.6), (7.10))

$$|D| = |D(\alpha, \beta)| = |D(\alpha_0 + h^1(t), \ \beta_0 + h^2(t))| \geqslant r(c)\,\xi_*{}^l, \ t_* \leqslant t \leqslant t^*$$

and, consequently, by virtue of (7.3), (7.4)

$$\rho(t) \geqslant r(c)\,\xi_*{}^l\,[(2l)!\,(\Gamma(\eta_*)^{2l-1}]^{-1}, \ t \in [t_*, t^*] \tag{7.13}$$

The upper bound for $\rho(t)$ follows from (5.2), (5.5), (7.12)

$$\rho(t) \leqslant \xi_* + \tau\Gamma(\eta_*)(1-\tau)^{-1} + |\omega + h(t)|\,\tau^l \leqslant \xi_* + 2\delta(c)(\tau(c))^{1/2} + c\,(\tau(c))^{1/2} \tag{7.14}$$

Here we have used the inequality $|\omega + h(t)| \leqslant c$, stipulated by (5.3) and by the choice of ω.

8. Let us complete the proof of the theorem. We consider the case $l = k$. Let us estimate the quantity

$$I(c) = \int_{t_*}^{t^*} \sigma(t, Y(t))\,|v_{\omega,c}(t)|^2 dt \tag{8.1}$$

Since, by the Lagrange theorem and by virtue of (2.2),

$$\sigma(t, Y(t)) \leqslant \sigma(t, x(t), y^{(1)}(t), \ldots, y^{(l-1)}(t)) + \mu_1(\eta(t))\,|x(t) - y(t)|\,\rho(t, X(t)) \leqslant \rho(t, X(t))\,(1 + \Gamma(\eta(t))\,|x(t) - y(t)|) \tag{8.2}$$

and since $|v_{\omega,c}(t)|^2 \leqslant |u(t)|^2 + |\omega|^2 (l!)^2 + 2l!\,|\omega|\,|u(t)|$,

$$I(c) \leqslant I_1(c) + I_2(c) + I_3(c) \tag{8.3}$$

$$I_1(c) = \int_{t_*}^{t^*} \sigma(t, Y(t))|u(t)|^2 dt$$

$$I_2(c) = c^2(l!)^2 \int_{t_*}^{t^*} \sigma(t, Y(t))dt \leqslant (cl!)^2 \tau(c)\Gamma(\Phi(F(\eta_*) + \varphi^*(\tau(c)))) \leqslant (cl!\delta(c))^2$$

$$I_3(c) = 2l!c \int_{t_*}^{t^*} \sigma(t, Y(t))\,|u(t)|\,dt$$

moreover, by virtue of the Cauchy-Buniakowski inequality

$$I_3(c) \leqslant 2\,(I_1(c))^{1/2}\,(I_2(c))^{1/2} \tag{8.4}$$

while by virtue of the inequalities (8.2), (7.14), (7.12), (3.2), (6.3)

$$I_1(c) \leqslant (1+\alpha(\eta_*, \tau(c))(\xi_* + (2\delta(c)+1)(\tau(c))^{1/2}))\, I_1^*(c) \leqslant \tag{8.5}$$
$$(1+\pi(\eta_*)\xi_* + 3\delta(c))\, I_1^*(c)$$

$$I_1^*(c) = \int_{t_*}^{t^*} \rho(t, X(t))\, |u(t)|^2 dt$$

We choose $\varepsilon_* = (\sigma - \rho) / a$. For the game point $z = \{t, X, Y\}$ we denote the function

$$\Psi(z) = |x^{(0)} - y^{(0)}|\, \pi(\eta(t, X, Y)) - 2\varepsilon_*$$

We set

$$c_n = c_0 n^{1/(2-4l)} \tag{8.6}$$

$$c_0 = \min\left\{3^{-1}\cdot 2^{-l}\left(\left(\frac{\varepsilon_*}{\Delta^* l!}\right)^2\left(2\sum_{n=1}^{\infty} n^{2l/(1-2l)}\right)^{-1}\right)^{1/4l}, \left(\frac{\varepsilon_*^2}{3}\right)^{1/(2l-1)}\right\} \tag{8.7}$$

Now, let the escape commence (see Sect. 1) at the point $z_0 = \{t_0, X_0, Y_0\}$, $|x_0^{(0)} - y_0^{(0)}| = \xi_0 \neq 0$. Let us assume that the escaper conducts the escape inductively by cycles, so that each mth cycle $(m \geqslant 1)$ consists of an interval of active escape of duration $\tau_m = \tau(c_m)$, if $m \geqslant 2$ and $\tau_1 = 0$, if $m = 1$, during which the escaper, having chosen the vector ω_m in accordance with the method of procedure used in Sect. 7, applies the special control $v(s) = v_{\omega_m, c_m}(s)$, and of a succeeding interval of passive escape during which the escaper applies the control $v(s) \equiv 0$ and whose duration θ_m is determined thus: $\theta_m = 0$ if $\Psi_m = \Psi(z(T_{m-1} + \tau_m)) \leqslant 0$ and θ_m is the smallest positive number for which $\Psi(z(T_{m-1} + \tau_m + \theta_m)) = 0$ if $\Psi_m > 0$. Here and subsequently

$$T_0 = t_0, \quad T_m = \sum_{k=2}^{m} (\tau_k + \theta_k), \quad m = 1, 2, \ldots$$

Then the following estimate (see (8.4) and the Cauchy-Buniakowski inequality) is valid for the escaper's control:

$$\int_{t_0}^{T_m} \sigma(t, Y(t))\, dt \leqslant \sum_{k=2}^{m} I(c_k) \leqslant \sum_{k=2}^{m} I_1(c_k) +$$
$$2\left(\sum_{k=2}^{m} I_1(c_k)\right)^{1/2}\left(\sum_{k=2}^{m} I_2(c_k)\right)^{1/2} + \sum_{k=2}^{m} I_2(c_k); \qquad m \geqslant 2$$

Since $(1 + \pi(\eta(T_k))\,\xi(T_k) + 3\delta(c_k)) \leqslant (1+\varepsilon_*)^2$, $k = 2, \ldots,$ from relation (8.5) follows

$$\sum_{k=2}^{m} I_1(c_k) \leqslant (1+\varepsilon_*)^2 \sum_{k=2}^{m} I_1^*(c_k) \leqslant (1+\varepsilon_*)^2 \rho^2 \tag{8.8}$$

the latter inequality is satisfied as soon as the pursuer, to whose behavior the escaper reacts, observes constraint (1.3). Further, by virtue of (8.6), (8.7)

$$\sum_{k=2}^{m} I_2(c_k) \leqslant (l!)^2 \sum_{k=2}^{m} c_k^2 \delta^2(c_k) < \varepsilon_*^2 \tag{8.9}$$

So that (see (8.8))

$$\int_{t_0}^{T_m} \sigma(t, Y(t))\, |v(t)|^2 dt \leqslant \left(\left(\sum_{k=2}^{m} I_1(c_k)\right)^{1/2} + \left(\sum_{k=2}^{m} I_2(c_k)\right)^{1/2}\right)^2 \leqslant \tag{8.10}$$
$$(1 + \varepsilon_*(\rho+1))^2 \leqslant \sigma^2$$

Let us prove that $T_m \to +\infty$ as $m \to \infty$. By contradiction, if $T_m \to b < +\infty$ as $m \to \infty$, then

$$b - t_0 \geqslant \sum_{k=1}^{\infty} \tau_k \geqslant \sum_{k=2}^{\infty} g(c_k)(\varepsilon(c_k))^{1/l} / p_k;\ p_k = \pi(\eta(T_{k-1})),\ k = 2, 3, \ldots$$

Since the series

$$\sum_{k=2}^{\infty} g(c_k)(\varepsilon(c_k))^{1/l} = (6\varepsilon^* c_0^{4l-2}) \sum_{k=2}^{\infty} \frac{1}{k}$$

diverges, the sequence p_k is unbounded, which by virtue of the continuity of the function $\pi(r)$ implies the unboundedness of the sequence $\eta(T_k)$. However, the latter is false, because if we set $v(t) \equiv u(t) \equiv 0$, $t \geqslant b$, then the functions $u(t)$ and $v(t)$ (already constructed on the half-interval $[t_0, b))$ are (see (8.10)) the player's controls, so that by virtue of the condition formulated in Sect. 1, the function $\eta(t) = \eta(t, X(t), Y(t))$, being an absolutely continuous function of the parameter $t \geqslant t_0$ [1], is bounded on $[t_0, b]$, which is a contradiction.

Passing to the limit as $m \to \infty$ in formula (8.10), we get that the constraint (1.4) imposed on the escaper's control is satisfied.

Let us obtain the estimate (2.3). On the passive segment the estimate follows from the definition of $\Psi(z)$

$$\xi(t) \geqslant 2\varepsilon_* / \pi(\eta(t)) \tag{8.11}$$

On the mth active segment $(m \geqslant 3)$ (see (7.13), (8.11), (6.3))

$$\xi(t) \geqslant r(c_m)\xi^l(T_{m-1})((2l)!(\Gamma(\eta(T_{m-1})))^{2l-1})^{-1} \geqslant \frac{r(c_m)(2\varepsilon_*)^l}{(2l)!\pi^l(\eta(T_{m-1}))(\Gamma(\eta(T_{m-1})))^{2l-1}} \geqslant \frac{r(c_m)(2\varepsilon_*)^l}{(2l)!}\gamma(\eta(t)) \tag{8.12}$$

where

$$\gamma(r) = \pi^{-l}\left(\Phi\left(\frac{F(r)+2\alpha^*}{3}\right)\right)\left(\Gamma\left(\Phi\left(\frac{F(r)+2\alpha^*}{3}\right)\right)\right)^{1-2l} < (\pi(r))^{-1}$$

if $\theta_{m-1} \neq 0$, and (see (7.13), (6.4), (6.3))

$$\xi(t) \geqslant \frac{r(c_m)(\varepsilon(c_{m-1}))^l}{(2l)!}\gamma(\eta(t)) \tag{8.13}$$

if $\theta_{m-1} = 0$.

In order to complete the proof it suffices to set

$$\varepsilon_m = \min_{m \geqslant 3} \{2\varepsilon_*,\ r(c_m)(2\varepsilon_*)^l / (2l)!,\ r(c_m)(\varepsilon(c_{m-1}))^l / (2l)!\} \tag{8.14}$$

and to note that (8.12) holds on the first active segment if $\theta_1 \neq 0$ and

$$\xi(t) \geqslant \frac{r(c_2)\xi_0^l}{(2l)!(\Gamma(\eta(t_0, X_0, Y_0)))^{2l-1}} \geqslant \frac{r(c_2)\xi_0^l}{(2l)!}\gamma(\eta(t)) \tag{8.15}$$

if $\theta_1 = 0$. So that $\varepsilon_2 = \min\{1,\ r(c_2)/(2l)!\}$, $\varepsilon = \varepsilon_*$.

9. In order to complete the theorem's proof for the case $l < k$, it suffices to set

$$\varepsilon_* = \sigma / a < \sigma \tag{9.1}$$

and, having noted that the estimate

$$I(c) \leqslant I_2(c) \leqslant (cl!\delta(c))^2$$

is valid for the quantity $I(c)$ (see (8.1)), to repeat verbatim the arguments in Sect. 8, having formally set in them $I_1(c_k) = 0$, $k = 2, 3, \ldots$, and having replaced in them the previous value of ε_* by the one given in formula (9.1). Here, as is easy to see, estimates (8.11) – (8.15) are preserved. The theorem is completely proved.

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Translated by N. H. C.

UDC 531.31

DYNAMIC SYSTEMS ARISING ON THE INVARIANT TORI OF THE KOWALEWSKA PROBLEM

PMM Vol. 39, № 1, 1975, pp. 24-29
V. V. KOZLOV
(Moscow)
(Received April 1, 1974)

We determine the gyration numbers of the dynamic systems arising on the two-dimensional invariant tori in Kowalewska's problem. We have shown that they equal the ratio of the periods of a hyperelliptic integral containing the Kowalewska polynomial. Using the general theorem on the reduction of equations on an n-dimensional torus, proved in the paper, the differential equations on the two-dimensional invariant tori mentioned are reduced by an invertible change of variables to the form $\varphi_i = \omega_i$ where $\omega_i = \text{const}$, $i = 1, 2$. We prove also that in the case of rapid rotations of the body the combined levels of the four first integrals of the problem consist of two tori; the dynamic systems arising on these tori are isomorphic.

1. Remarks on the topological properties of the combined levels of first integrals. The Euler-Poisson equations of the problem of the motion of a heavy rigid body around a fixed point form an analytic system of differential equations defined in R^6 $\{x : pqr\gamma_1\gamma_2\gamma_3\}$. There is an integral invariant in this system, whose density $M(x) \equiv 1$ (i.e. the phase volume is invariant relative to a one-parameter group g^t of shifts along the trajectories of the Euler-Poisson equations). These equations always have three algebraic first integrals: the energy integral (H), the area integral (L) and the geometric integral (Γ). If the rigid body is a Kowalewska top, then there exists a fourth algebraic integral K.

By E we denote the following set:

$$E = \{x : H = 6h,\ L = 2l,\ \Gamma = 1,\ K = k^2\}\ (E \subset R^6)$$

It is compact, since the set $\{H = 6h,\ \Gamma = 1\}$ is bounded in R^6 and E is closed.